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SELF-SIMILAR PROBLEMS OF PROPAGATION OF TANGENTIAL RUPTURE CRACKS

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B. V. Kostrov
(Moscow)

ABSTRACT

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An examination is made of the plane and spatial /889*
problems of unsteady crack propagation in a medium subjected to uniform shear. The plane problem is completely analogous to the problem of Broberg (Ref.1) regarding a normal rupture crack, but it is solved by much simpler methods. The concurrent study of plane and spatial cases also has the advantage that several of the intermediate results derived for the plane problem provide a basis for solving the spatial case.

Author

The axisymmetric problem regarding propagation of a normal rupture crack (the spatial analog of Broberg's problem) was solved by the author in (Ref. 2). In contrast to this problem, the problem to be solved in the present study will not be axisymmetric, but a generalization of the method employed in (Ref. 2) makes it possible to formulate a precise solution for the problem at hand. It is thus assumed that the surface of the crack has the form of a circular disc, i.e., the propagation velocity of the crack does not depend on direction. It is shown that the latter assumption cannot, generally speaking, be put into

* Note: Numbers in the margin indicate pagination in the original foreign text.

effect, but one can formulate an initial stress magnitude at which it will be valid. For all other values of initial stress, the solution obtained can be regarded as approximative.

1. Formulation of the Problem

a) Plane Case. A uniform and isotropic elastic medium with the shear modulus μ and with propagation velocities of the longitudinal and transverse waves a and b occupies infinite space. For $t < 0$, only one component of the stress tensor $\tau_{xz}{}^{\circ} = \tau^{\circ}$ differs from zero. At the moment of time $t = 0$, a crack is formed along the y-axis, which is then propagated in the plane $z = 0$, so that the elastic disturbances which are thus formed do not depend on the coordinates y and are polarized in the xz-plane. The crack propagation velocity is assumed to be constant, and is designated by α. The crack location is shown in Figure 1. Tangential stresses must disappear at the crack surface, i.e., disturbances produced by development of the crack must satisfy the condition

$$\tau_{xz} = -\tau^{\circ} \quad \text{for} \quad z = 0, \ |x| \leqslant \alpha t$$

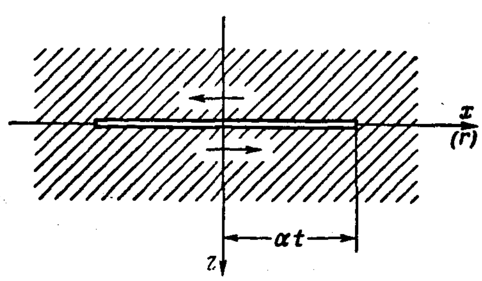

Figure 1

It can be shown that the displacement vector in the disturbance must be antisymmetric with respect to the plane z = 0. Thus, the tangential displacement and normal stress are odd functions of z, and thus the following boundary conditions hold at z = 0: /890

$$
\begin{aligned}
\tau_{xz} &= -\tau^0 && \text{for} \quad z = 0,\ |x| < \alpha t \\
\sigma_z &= 0 && \text{for} \quad z = 0,\ -\infty < x < \infty \\
u_x &= 0 && \text{for} \quad z = 0,\ |x| > \alpha t
\end{aligned}
\tag{1.1}
$$

Since the crack and the elastic disturbances related to it are not present for $t \leqslant 0$, the initial conditions have the form ($u = \{u_x, u_z\}$ - displacement vector)

$$u = 0,\ \dot{u} = v = 0 \text{ for } t = 0$$

The dot employed here designates time derivative.

A condition must be imposed on the solution behavior in the vicinity of the crack edge, besides the boundary and initial conditions. Just as in the case of a normal rupture crack, we can assume that the crack edge is surrounded by a region in which plastic deformation of the medium occurs. The dimensions of this region increase at constant velocity which is proportional to the crack propagation velocity α, but always remain much smaller than the dimensions of the crack itself. In this way, this plastic region can be regarded as infinitely small. Let us also assume that the work expended on forming the crack is proportional to the volume of the plastic region, so that the corresponding output can be written in the form

$$w = 2\alpha^2 tC \tag{1.2}$$

where C is the constant which is independent of α. This output must equal the energy flux through the surface surrounding the crack edge and located at an infinitely small distance from it. We thus obtain the requisite condition in the form

$$\lim_{\delta \to 0} \int_{l_\delta} \mathbf{t}_n \mathbf{v} \, dl = \alpha^2 t C \qquad (1.3)$$

The contour l_δ encompasses one of the crack edges and is located at the distance δ from it.

In particular, it follows from (1.3) that the components of stress and velocity must increase as the crack edge is approached, as $\delta^{-\frac{1}{2}}$. It can be readily seen that the components of stress and velocity are uniform functions of the coordinates and of time measured from zero. Consequently, close to the crack edge they are proportional to $\sqrt{t/\delta}$.

In physical terms, it is clear that this plastic region must reach a certain stationary size in time. Then the right part in (1.3) becomes constant (and not proportional to time), so that the stresses in the vicinity of the crack edge must become proportional to $\sqrt{1/\delta}$ in time, and not to $\sqrt{t/\delta}$, i.e., the self-similar nature of the problem must be disturbed or, in other words, the assumption regarding the uniformity of the crack propagation velocity. Thus, this formulation of the problem is advantageous only for the initial period of crack development.

b) Spatial Case. The initial stress state of the medium is the same as in the plane case, but the crack propagation is now initiated at the origin. It is assumed that the crack propagation velocity is constant and does not depend on direction, so that the crack surface is determined by the following relationships for $t > 0$:

$$z = 0, \quad 0 \leqslant r < \alpha t$$

at the cylindrical coordinates r, ϕ, z. Just as in the plane case, we reduce the problem at hand to the boundary problem for half-space $z \geqslant 0$ with the boundary conditions

$$\tau_{rz} = -\tau^\circ \cos\varphi, \qquad \tau_{\varphi z} = \tau^\circ \sin\varphi \text{ for } \quad z = 0,\ r < \alpha t$$
$$\sigma_z = 0 \text{ for } z = 0,\ 0 \leqslant r < \infty;\ u_r = u_\varphi^\cdot = 0; \text{ for } z = 0,\ r > \alpha t \qquad (1.4)$$

and with the initial conditions

$$u = 0 \qquad u^\cdot \equiv v = 0, \text{ for } t = 0 \qquad (1.5)$$

An additional condition can be written in the form

$$\lim_{\delta\to 0} \int_{S_\delta} t_n v\, dS = 2\pi\alpha^3 t^2 C \qquad (1.6)$$

where S_δ is the toroidal surface surrounding the crack edge and located at the distance δ from it.

All the statements pertaining to the plane problem remain in force for the spatial case, but one fact now stands out. Strictly speaking, condition (1.6) must be formulated for the vicinity of each point on the crack edge, but it then appears that in the general case α, which is determined from this condition depends on direction, i. e., the form of the crack must differ from a circular form. Unfortunately, at the present time there is no known method for solving the problem for the case when the crack edge is an arbitrary curve. Thus, a certain effective value for the crack propagation velocity is obtained from condition (1.6). It must be pointed out that, although this formulation of the problem is not valid for determining the form of the crack edge, its solution must yield a correct description of the elastic wave fields at large distances from the crack. This is of basic importance when applying this solution to seismology. The last part of Section 4 will give a value for initial stress, at which the crack form will be circular, even if local fulfillment of an additional condition is required (at this initial stress value, the integrand in (1.6) does not depend on the angle ϕ).

2. Functionally-Invariant Solutions

In both of the formulated problems, the components of the stress tensor and the velocity vector are uniform functions of the coordinates and of time measured from zero. This makes it possible to employ the method of functionally-invariant solutions given by Smirnov-Sobolev[1]. If this method is employed, the plane problem can be readily solved for half-space $z \geqslant 0$ with the displacement vector polarized in the xz-plane; this solution satisfies the following condition at the boundary

$$\sigma_z = 0 \quad \text{for } z=0, \ -\infty < x < \infty \tag{2.1}$$

Omiting the details, we can write this solution as follows:

$$\begin{aligned} \dot{u}_x \equiv v_x = v_x^{(1)} + v_x^{(2)}, &\qquad v_x^{(1,2)} = \operatorname{Re} V_x^{(1,2)} (\vartheta^{(1,2)}) \\ \dot{u}_z \equiv v_z = v_z^{(1)} + v_k^{(2)}, &\qquad v_z^{(1,2)} = \operatorname{Re} V_z^{(1,2)} (\vartheta^{(1,2)}) \\ \sigma_z = \sigma_z^{(1)} + \sigma_z^{(2)}, &\qquad \sigma_z^{(1,2)} = \operatorname{Re} \Sigma_z^{(1,2)} (\vartheta^{(1,2)}) \\ \tau_{xz} = \tau_{xz}^{(1)} + \tau_{xz}^{(2)}, &\qquad \tau_{xz}^{(1,2)} = \operatorname{Re} T_{xz}^{(1,2)} (\vartheta^{(1,2)}) \end{aligned} \tag{2.2}$$

[1] See Chapter XII, in the Russian translation of the book (Ref. 3).

Here $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are determined from the equations

$$\delta^{(1)} \equiv t - \vartheta^{(1)}x - z\sqrt{a^{-2} - \vartheta^{(1)2}} = 0, \quad \delta^{(2)} \equiv t - \vartheta^{(2)}x - z\sqrt{b^{-2} - \vartheta^{(2)2}} = 0 \qquad (2.3)$$

and the functions in (2.2) under the real part are expressed by /892

means of a single unknown function $V(\vartheta)$ by the relationships

$$V_x^{(1)'}(\vartheta) = 2b^2\vartheta^2 V'(\vartheta), \qquad V_x^{(2)'}(\vartheta) = (1 - 2b^2\vartheta^2)\, V'(\vartheta) \qquad (2.4)$$
$$V_z^{(1)'}(\vartheta) = 2b^2\vartheta\sqrt{a^{-2} - \vartheta^2}\, V'(\vartheta)$$
$$V_z^{(2)'}(\vartheta) = -\vartheta\,(1 - 2b^2\vartheta^2)\,(b^{-2} - \vartheta^2)^{-1/2}\, V'(\vartheta)$$
$$T_{xz}^{(1)'}(\vartheta) = -4\mu b^2\vartheta^2\sqrt{a^{-2} - \vartheta^2}\, V'(\vartheta), \quad T_{xz}^{(2)'}(\vartheta) = \frac{-4\mu b^2\,(\vartheta^2 - {}^1/_2 b^{-2})^2}{\sqrt{b^{-2} - \vartheta^2}}\, V'(\vartheta)$$
$$\Sigma_z^{(1)'}(\vartheta) = -\Sigma_z^{(2)'}(\vartheta) = -2\mu\vartheta\,(1 - 2b^2\vartheta^2)\, V'(\vartheta)$$

One solution of the plane problem, in which the displacement vector is parallel to the y-axis, is still required to solve the spatial problem. This solution is determined by the relationships

$$u_y^{\cdot} \equiv v_y = v_y^{(2)}, \quad v_y^{(2)} = \operatorname{Re} V_1(\vartheta^{(2)})$$
$$\tau_{yz} = \tau_{yz}^{(2)}, \quad \tau_{yz}^{(2)} = \operatorname{Re} T_{yz}^{(2)}(\vartheta^{(2)}) \quad (2.5) \qquad (2.5)$$
$$T_{yz}^{(2)'}(\vartheta) = -\mu\sqrt{b^{-2} - \vartheta^2}\, V_1'(\vartheta)$$

Instead of the real parts, one can take the imaginary parts of the corresponding functions in formulas (2.2) and (2.5).

3. Solution of the Plane Problem

The relationships in the preceding section express the derivatives of the desired functions by the derivative of the function $V(\vartheta)$. Their primatives must be determined so that the initial conditions are fulfilled. It can be readily seen that integration must be carried out with respect to the contours shown in Figure 2 in order to do this, so that

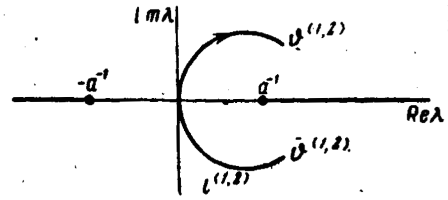

Figure 2.

$$v_x^{(1)} = \frac{1}{2i} \int\limits_{l^{(1)}} V_x^{(1)\prime}(\lambda)\, d\lambda = -ib^2 \int\limits_{l^{(1)}} \lambda^2 V'(\lambda)\, d\lambda \tag{3.1}$$

$$v_x^{(2)} = \frac{1}{2i} \int\limits_{l^{(2)}} V_x^{(2)}(\lambda)\, d\lambda = \frac{1}{2i} \int\limits_{l^{(2)}} (1 - 2b^2\lambda^2)\, V'(\lambda)\, d\lambda \quad \text{etc.}$$

The initial conditions will be fulfilled, if it is required that $V'(\lambda)$ be regular for $-a^{-1} < \mathrm{Re}\lambda < a^{-1}$, and if the radicals $\sqrt{a^{-2} - \lambda^2}$ and $\sqrt{b^{-2} - \lambda^2}$ are transformed so that they are positive for $\lambda = 0$, carrying out branch cuts from $-a^{-1}$ to $-\infty$ and from a^{-1} to ∞ along the real axis.

For $z = 0$ the functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ assume the same values $\vartheta^{(1)} = \vartheta^{(2)} = \vartheta \equiv t/x$, and as a result we obtain

$$v_x = \frac{1}{2i} \int\limits_{l} V'(\lambda)\, d\lambda, \qquad \tau_{xz} = 2i\mu b^2 \int\limits_{l} \frac{R(\lambda^2)}{\sqrt{b^{-2} - \lambda^2}} V'(\lambda)\, d\lambda \quad \text{for } z = 0 \tag{3.2}$$

where l is the contour shown in Figure 3 and

$$R(\lambda^2) = (\lambda^2 - {}^1/_2\, b^{-2})^2 + \lambda^2 \left(\sqrt{a^{-2} - \lambda^2}\sqrt{b^{-2} - \lambda^2}\right)$$

Expressions (3.2) must satisfy the conditions (1.1). Analyzing (3.2), and from these conditions as well as the requisite solution behavior in the vicinity of the crack edge, we obtain /893

$$V'(\lambda) = \frac{A}{(\alpha^{-2} - \lambda^2)^{3/2}} \quad \text{for} \quad 0 < \alpha < c \tag{3.3}$$

$$V'(\lambda) = \frac{A}{(\alpha^{-2} - \lambda^2)^{3/2}} + \frac{B}{(c^{-2} - \lambda^2)(\alpha^{-2} - \lambda^2)^{1/2}} \quad \text{for} \quad c < \alpha < b \tag{3.4}$$

Here c is the velocity of Rayleigh waves ($R(c^{-2}) = 0$). In the case $b < \alpha < a$ it can be stated that there is no solution which has the requisite order of increase close to the crack edge. It can be shown that the integral in the right part of (1.3) in the case $c < \alpha < b$ is negative, i.e., in this case, an additional condition cannot be fulfilled. From this point on, we shall assume that $\alpha < c$. Utilizing (3.3), we obtain the displacement of the crack boundary from the first expression (3.2):

$$u_x = \alpha A \sqrt{\alpha^2 t^2 - x^2} \quad \text{for } z = 0,\ |x| < \alpha t \tag{3.5}$$

The second of the relationships (3.2) gives the equation for determining A. In order to do this, in the case of $|\vartheta| < \alpha^{-1}$ let us deform the contour l so that it coincides with the imaginary axis. This can be done, since the integrand is regular outside of the branch cuts from $\pm a^{-1}$ to $\pm \alpha^{-1}$, and decreases rapidly to infinity. Then from (1.1) and (3.2), we have

$$\tau^\circ = 4\mu b^2 A \int_0^\infty \frac{(\lambda^2 + \frac{1}{2}b^{-2})^2 - \lambda^2 \sqrt{a^{-2} + \lambda^2}\sqrt{b^{-2} + \lambda^2}}{\sqrt{b^{-2} + \lambda^2}\,(\alpha^{-2} + \lambda^2)^{3/2}}\, d\lambda$$

Let us designate the integral in this formula by $I(\alpha)$. We then have

$$A = \frac{\tau^\circ}{4\mu b^2 I(\alpha)} \tag{3.6}$$

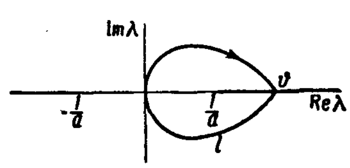

Figure 3.

We must now study the solution behavior close to the crack edge and must fulfill the additional condition (1.3). When the point (x, z) approaches the crack edge, the ends of the contours $l^{(1)}$ and $l^{(2)}$ approach the point $\lambda = \alpha^{-1}$ (as the right crack edge is approached). Using the Laplace method, we can readily obtain the first terms of asymptotic expansions of the velocity and stress components in the form

$$v_x^{(1)} \approx 2b^2 A\ (\alpha t / 2\delta)^{1/2}\, \mathrm{Im}\, f^{(1)}(\psi), \qquad v_x^{(2)} \approx (\alpha^2 - 2b^2)\,(\alpha t/2\delta)^{1/2}\, \mathrm{Im}\, f^{(2)}(\psi)$$
$$v_z^{(1)} \approx 2b^2 \sqrt{1 - \alpha^2 a^{-2}}\, A\ (\alpha t / 2\delta)^{1/2}\, \mathrm{Re}\, f^{(1)}(\psi)$$
$$v_z^{(2)} \approx \frac{\alpha^2 - 2b^2}{\sqrt{1-\alpha^2 b^{-2}}}\, A \left(\frac{\alpha t}{2\delta}\right)^{1/2} \mathrm{Re}\, f^{(2)}(\psi)$$
$$\sigma_z^{(1)} \approx 2\mu\alpha^{-1}\,(2b^2 - \alpha^2)\, A\ (\alpha t / 2\delta)^{1/2}\, \mathrm{Im}\, f^{(1)}(\psi)$$
$$\sigma_z^{(2)} \approx 2\mu\alpha^{-1}\,(\alpha^2 - 2b^2)\, A\ (\alpha t / 2\delta)^{1/2}\, \mathrm{Im}\, f^{(2)}(\psi)$$
$$\tau_{xz}^{(1)} \approx 4\mu b^2 \alpha^{-1} \sqrt{1 - \alpha^2 a^{-2}}\, A\ (\alpha t / 2\delta)\ \mathrm{Re}\, f^{(1)}(\psi)$$
$$\tau_{xz}^{(2)} \approx 4\mu b^2 \frac{(1 - \alpha^2/2b^2)^2}{\alpha\sqrt{1-\alpha^2 b^{-2}}}\, A \left(\frac{\alpha t}{2\delta}\right)^{1/2} \mathrm{Re}\, f^{(2)}(\psi) \tag{3.7}$$

where /894

$$|x| = \alpha t + \delta \cos\psi,\ z = \delta \sin\psi,\quad f^{(1)}(\psi) = (\cos\psi - i \sin\psi \sqrt{1 - \alpha^2 a^{-2}})^{-1/2}$$
$$f^{(2)}(\psi) = (\cos\psi - i \sin\psi \sqrt{1 - \alpha^2 b^{-2}})^{-1/2}$$

We can now readily calculate the limits of the integral in the left part of (1.3). Employing (3.6), we obtain as a result

$$\frac{\pi \tau^{\circ 2}}{16\mu b^2 [I(\alpha)]^2 \sqrt{1-\alpha^2 b^{-2}}}\left[\left(1 - \frac{\alpha^2}{a^2}\right)^{1/2}\left(1 - \frac{\alpha^2}{b^2}\right)^{1/2} - \left(1 - \frac{\alpha^2}{2b^2}\right)\right] = C \tag{3.8}$$

This equation determines the crack propagation velocity α as a function of the load τ°.

4. Solution of the Spatial Problem

The solution of the spatial problem can be reduced to the plane problem, by the same method as was employed in the study (Ref. 2) for the axisymmetric problem of a normal rupture crack. The case under consideration differs from the latter, due to the fact that the problem is not axisymmetric.

Let us introduce the Cartesian system of coordinates x, y, z, which is dependent on the parameter ω and which is related to the basic polar system r, φ, z by the relationships

$$x = r\cos(\varphi - \omega), \qquad y = r\sin(\varphi - \omega), \qquad z = z$$

and let us form a superposition of the plane solution of the following type:

$$\mathbf{u} = \int_{-\pi}^{\pi} [\mathbf{u}_1(x, z, t)\cos\omega + \mathbf{u}_2(x, z, t)\sin\omega]\, d\omega$$

where $u_1(x, z, t)$ is determined from (2.2), and $u_2(x, z, t)$ - from (2.5). It can be readily seen that the vector obtained satisfies the equations of motion, the condition

$$\sigma_z = 0 \quad \text{for } z = 0,\ 0 \leqslant r < \infty$$

and has the requisite dependence on φ. Replacing the variable $\Omega = \varphi - \omega$ and utilizing the equations (2.2) - (2.5), we obtain

$$u_r^{\cdot} \equiv v_r = v_r^{(1)} + v_r^{(2)}, \qquad v_r^{(1)} = \cos\varphi \int_{-\pi}^{\pi} V_x^{(1)}(\vartheta^{(1)}) \cos^2\Omega\, d\Omega$$

$$v_r^{(2)} = \cos\varphi \int_{-\pi}^{\pi} [V_x^{(2)}(\vartheta^{(2)})\cos^2\Omega - V_1(\vartheta^{(2)})\sin^2\Omega]\, d\Omega$$

$$u_\varphi^{\cdot} \equiv v_\varphi = v_\varphi^{(1)} + v_\varphi^{(2)}, \qquad v_\varphi^{(1)} = -\sin\varphi \int_{-\pi}^{\pi} V_x^{(1)}(\vartheta^{(1)}) \sin^2\Omega\, d\Omega$$

$$v_\varphi^{(2)} = -\sin\varphi \int_{-\pi}^{\pi} [V_x^{(2)}(\vartheta^{(2)})\sin^2\Omega - V_1(\vartheta^{(2)})\cos^2\Omega]\, d\Omega$$

$$u_z^{\cdot} \equiv v_z = v_z^{(1)} + v_z^{(2)}, \qquad v_z^{(1,2)} = \cos\varphi \int_{-\pi}^{\pi} V_z^{(1,2)}(\vartheta^{(1,2)}) \cos\Omega\, d\Omega$$

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$$\sigma_z = \sigma_z^{(1)} + \sigma_z^{(2)}, \qquad \sigma_z^{(1,2)} = \cos\varphi \int_{-\pi}^{\pi} \Sigma_z^{(1,2)}(\vartheta^{(1,2)}) \cos\Omega\, d\Omega$$

$$\tau_{rz} = \tau_{rz}^{(1)} + \tau_{rz}^{(2)}, \qquad \tau_{rz}^{(1)} = \cos\varphi \int_{-\pi}^{\pi} T_{xz}^{(1)}(\vartheta^{(1)}) \cos^2\Omega\, d\Omega \qquad (4.1)$$

$$\tau_{rz}^{(2)} = \cos\varphi \int_{-\pi}^{\pi} [T_{xz}^{(2)}(\vartheta^{(2)}) \cos^2\Omega - T_{yz}^{(2)}(\vartheta^{(2)}) \sin^2\Omega]\, d\Omega$$

$$\tau_{\varphi z} = \tau_{\varphi z}^{(1)} + \tau_{\varphi z}^{(2)}, \qquad \tau_{\varphi z}^{(1)} = -\sin\varphi \int_{-\pi}^{\pi} T_{xz}^{(1)}(\vartheta^{(1)}) \sin^2\Omega\, d\Omega$$

$$\tau_{\varphi z}^{(2)} = -\sin\varphi \int_{-\pi}^{\pi} [T_{xz}^{(2)}(\vartheta^{(2)}) \sin^2\Omega - T_{yz}^{(2)}(\vartheta^{(2)}) \cos^2\Omega]\, d\Omega$$

In these expressions, the functions $\vartheta^{(1,2)}$ are determined from the equations

$$\delta^{(1)} \equiv t - \vartheta^{(1)} r\cos\Omega - z\sqrt{a^{-2} - \vartheta^{(1)2}} = 0$$
$$\delta^{(2)} \equiv t - \vartheta^{(2)} r\cos\Omega - z\sqrt{b^{-2} - \vartheta^{(2)2}} = 0$$

and the functions under the integrals are expressed by the two unknown functions $V(\vartheta)$ and $V_1(\vartheta)$, according to (2.2) and (2.5). It can be readily seen that $V(\vartheta)$ and $V_1(\vartheta)$ can be regarded as odd functions of ϑ, since the components in (4.1) - corresponding to the odd parts of these functions - disappear identically. Let us use the notation

$$F(\vartheta^2) = V(\vartheta), \qquad F_1(\vartheta^2) = V_1(\vartheta)$$

Taking the fact into account that $\vartheta^{(1)} = \vartheta^{(2)} = \vartheta \equiv t / r\cos\Omega$ in the case of z = 0, we can obtain the following expressions from (4.1):

$$\frac{r}{2\cos\varphi} v_r^{\cdot} = \operatorname{Re} \int_{l_\nu} \left[\frac{\nu_0}{\nu} F'(\nu) - \left(1 - \frac{\nu_0}{\nu}\right) F_1'(\nu)\right] \frac{d\nu}{\sqrt{\nu - \nu_0}}, \quad \nu_0 = \frac{t^2}{r^2}$$

$$-\frac{r}{2\sin\varphi} v_\varphi^{\cdot} = \operatorname{Re} \int_{l_\nu} \left[\left(1 - \frac{\nu_0}{\nu}\right) F'(\nu) - \frac{\nu_0}{\nu} F_1'(\nu)\right] \frac{d\nu}{\sqrt{\nu - \nu_0}}, \quad \nu = \vartheta^2$$

$$-\frac{r}{2\mu\cos\varphi} \tau_{rz}^{\cdot} = \operatorname{Re} \int_{l_\nu} \left[\frac{4b^2 R(\nu)}{\sqrt{b^{-2} - \nu}} F'(\nu) \frac{\nu_0}{\nu} - \left(1 - \frac{\nu_0}{\nu}\right)\sqrt{b^{-2} - \nu}\, F_1'(\nu)\right] \frac{d\nu}{\sqrt{\nu - \nu_0}} \tag{4.2}$$

$$\frac{r}{2\mu\sin\varphi}\tau_{\varphi z} = \operatorname{Re}\int\limits_{l_\nu}\left[\frac{4b^2T(\nu)}{\sqrt{b^{-2}-\nu}}\left(1-\frac{\nu_0}{\nu}\right)F'(\nu) - \right.$$
$$\left. -\frac{\nu_0}{\nu}\sqrt{b^{-2}-\nu}\,F'_1(\nu)\right]\frac{d\nu}{\sqrt{\nu-\nu_0}} \quad \text{for } z = 0$$

The integration contour l_ν is shown in Figure 4. In order to fulfill the initial conditions, expressions (4.2) must disappear for $\nu_0 < a^{-2}$. This will be fulfilled, if $F'(\nu)$ and $F_1'(\nu)$ - which are regular outside of the branch cut from a^{-2} to ∞ - satisfy the con- /896
dition

$$F'(0) = -F_1'(0) \tag{4.3}$$

and decrease to infinity more rapidly than ν^{-1}. In view of the boundary conditions, the first two of the expressions in (4.2) must disappear for $\nu_0 < \alpha^{-2}$. In order to do this, $F'(\nu)$ and $F_1'(\nu)$ must be regular for Re $\nu < \alpha^{-2}$. On the other hand, the last two expressions in (4.2) must disappear for $\nu_0 > \alpha^{-2}$. In order to do this, the integrands must be regular for Re $\nu > \nu_0 > \alpha^{-2}$. We find $F'(\nu)$ and $F_1'(\nu)$ from these conditions and the requisite stress behavior in the vicinity of the crack edge. It can be shown that, just as in the plane problem, an additional condition can be fulfilled only for $\alpha < c$. The functions $F'(\nu)$ and $F_1'(\nu)$ satisfy the requisite conditions in the following form:

$$F'(\nu) = -F_1'(\nu) = \frac{A}{(\alpha^{-2}-\nu)^2}$$

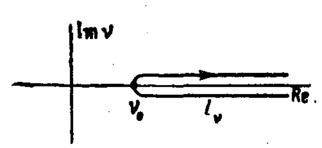

Figure 4.

where A is an indefinite constant.

Similarly to expression (4.2), we can obtain the following for $z = 0$:

$$\begin{aligned}
v_r &= \sqrt{\nu_0}\cos\varphi \operatorname{Re}\int_{l_\nu}\left[\frac{\nu_0}{\nu}F(\nu)-\left(1-\frac{\nu_0}{\nu}\right)F_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}\\
v_\varphi &= -\sqrt{\nu_0}\sin\varphi \operatorname{Re}\int_{l_\nu}\left[\left(1-\frac{\nu_0}{\nu}\right)F(\nu)-\frac{\nu_0}{\nu}F_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}\\
\tau_{rz} &= -\sqrt{\nu_0}\cos\varphi \operatorname{Re}\int_{l_\nu}\left[\frac{\nu_0}{\nu}G(\nu)-\left(1-\frac{\nu_0}{\nu}\right)G_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}\\
\tau_{\varphi z} &= \sqrt{\nu_0}\sin\varphi \operatorname{Re}\int_{l_\nu}\left[\left(1-\frac{\nu_0}{\nu}\right)G(\nu)-\frac{\nu_0}{\nu}G_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}
\end{aligned}\tag{4.4}$$

where

$$F(\nu)=\int_0^\nu F'(\lambda)\,d\lambda,\qquad F_1(\nu)=\int_0^\nu F_1'(\lambda)\,d\lambda,\qquad G(\nu)=\mu\int_0^\nu \frac{4b^2R(\lambda)}{\sqrt{b^{-2}-\lambda}}F'(\lambda)\,d\lambda$$

$$G_1(\nu)=\mu\int_0^\nu \sqrt{b^{-2}-\lambda}F'_1(\lambda)\,d\lambda$$

The lower limit here is selected as being equal to zero, in order that the integrands in (4.4) be regular for $\nu = 0$, which is necessary in order that the initial conditions be fulfilled.

The functions $F(\nu)$ and $F_1(\nu)$ can be readily calculated; as a result, we obtain

$$F(\nu)=-F_1(\nu)=\frac{\alpha^2A\nu}{\alpha^{-2}-\nu}$$

and from (4.4) we find

$$v_r=-2\pi\cos\varphi\frac{\alpha^2At}{\sqrt{\alpha^2t^2-r^2}},\qquad v_\varphi=2\pi\sin\varphi\frac{\alpha^2At}{\sqrt{\alpha^2t^2-r^2}}\ \text{ for } z=0,\ r<\alpha t \tag{4.5}$$

We can transform the expressions for $G(\nu)$ and $G_1(\nu)$ in the following way:

$$G(\nu)=4\mu b^2\int_0^{-\infty}\frac{R(\nu)}{\sqrt{b^{-2}-\nu}}F'(\nu)\,d\nu+4\mu b^2\int_{\nu\infty}^{\nu}\frac{R(\lambda)}{\sqrt{b^{-2}-\nu}}F'(\nu)\,d\lambda=M+G^*(\nu)$$

$$G_1(\nu)=\mu\int_0^{-\infty}\sqrt{b^{-2}-\nu}F_1'(\nu)\,d\nu+\mu\int_{\nu\infty}^{\nu}\sqrt{b^{-2}-\lambda}F_1'(\lambda)\,d\lambda=M_1+G_1^*(\nu)$$

The terms in (4.4), which correspond to $G^*(\nu)$ and $G_1^*(\nu)$, disappear for $\nu_0 > \alpha^{-2}$, since the integrands containing these functions are regular for $\mathrm{Re}\ \nu > \nu_0 > \alpha^{-2}$. We thus have

$$\tau_{rz} = -(M - M_1)\pi\cos\varphi, \qquad \tau_{\varphi z} = (M - M_1)\pi\sin\varphi \quad \text{for} \quad z = 0,\ r < \alpha t$$

Comparing these value with the boundary conditions (1.4), we obtain

$$(M - M_1)\pi = \tau^\circ$$

or

$$\tau^\circ = -\mu\pi A \int_0^\infty \left[4b^2 \frac{(\nu + 1/2 b^{-1})^2 - \nu\sqrt{a^{-2}+\nu}\sqrt{b^{-2}+\nu}}{\sqrt{b^{-2}+\nu}} + \sqrt{b^{-2}+\nu}\right] \frac{d\nu}{(\alpha^{-2}+\nu)^2} \qquad (4.6)$$

We shall designate the integral in this equation by $I_1(\alpha)$, and we shall set $A_1 = -\pi A$. We then have

$$A_1 = \frac{\tau^\circ}{\mu I_1(\alpha)} \qquad (4.7)$$

Integrating expression (4.5) with respect to time, we now obtain

$$u_r = 2A_1\alpha\cos\varphi\sqrt{\alpha^2t^2 - r^2}, \quad u_\varphi = -2A_1\alpha\sin\varphi\sqrt{\alpha^2t^2 - r^2} \quad \text{for } z = 0,\ r \leqslant \alpha \qquad (4.8)$$

or in Cartesian coordinates,

$$u_x = 2\alpha A_1\sqrt{\alpha^2t^2 - r^2}, \qquad u_y = 0 \quad \text{for } z = 0,\ r \leqslant \alpha t \qquad (4.9)$$

Here $x = r\cos\varphi$, $y = r\sin\varphi$. Thus, the displacement direction of the crack edges coincides with the direction of initial stress.

Using the method described in the work (Ref. 2), we can obtain the asymptotic expressions for velocity and stress close to the crack edge

$$
\begin{aligned}
v_r^{(1)} &\approx 2b^2 A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(1)}(\psi),\\
v_r^{(2)} &\approx (\alpha^2 - 2b^2) A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(2)}(\psi)\\
v_\varphi^{(1)} &\approx O(1), \qquad v_\varphi^{(2)} \approx -\alpha^2 A_1 \sin\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(2)}(\psi)\\
v_z^{(1)} &\approx 2b^2\sqrt{1-\alpha^2 a^{-2}}\,A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(1)}(\psi)\\
v_z^{(2)} &\approx (\alpha^2 - 2b^2)(1-\alpha^2 b^{-2})^{-1/2} A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(2)}(\psi)\\
\sigma_z^{(1)} &\approx -2\mu\alpha^{-1}(\alpha^2 - 2b^2) A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(1)}(\psi)\\
\sigma_z^{(2)} &\approx 2\mu\alpha^{-1}(\alpha^2 - 2b^2) A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(2)}(\psi)\\
\tau_{rz}^{(1)} &\approx -4\mu b^2\alpha^{-1}\sqrt{1-\alpha^2 a^{-2}}\,A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(1)}(\psi)\\
\tau_{rz}^{(2)} &\approx \mu b^{-2}\alpha^{-1}(1-\alpha^2 b^{-2})^{-1/2}(\alpha^2 - 2b^2)^2 A_1 \cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(2)}(\psi)\\
\tau_{\varphi z}^{(1)} &\approx O(1), \qquad \tau_{\varphi z}^{(2)} \approx \mu\alpha\sqrt{1-\alpha^2 b^{-2}}\,A_1 \sin\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(2)}(\psi)
\end{aligned}
\tag{4.10}
$$

$r = \alpha t + \delta\cos\psi, \quad z = \delta\sin\psi,$ /898

where the functions $f^{(1)}(\psi)$ and $f^{(2)}(\psi)$ are the same as in (3.7). Employing these expressions, we can obtain the following relationship from the condition (1.6):

$$
\mu A_1 \int_0^{2\pi} \left\{ \frac{4b^2\left[\sqrt{1-\alpha^2 a^{-2}}\sqrt{1-\alpha^2 b^{-2}} - (1-\alpha^2/2b^2)^2\right]}{\sqrt{1-\alpha^2 b^{-2}}}\cos^2\varphi + \right.
$$
$$
\left. + \alpha^2\sqrt{1-\alpha^2 b^{-2}}\sin^2\varphi \right\} d\varphi = C \tag{4.11}
$$

It can thus be seen that - as was noted when the problem was formulated - the velocity α must be, strictly speaking, a function of φ, since in the general case the integrand depends on ϕ with constant α. However, there is one value of α - namely,

$$
\alpha = \alpha_1 = b\left(1 - \frac{a^2}{9a^2 - 16b^2}\right)^{1/2} < c \tag{4.12}
$$

at which the integrand in (4.11) is constant, since in this case the coefficients coincide for $\cos^2\varphi$ and for $\sin^2\varphi$. With this value of α, the crack will have a circular form. The corresponding magnitude of initial stress is obtained from (4.7), (4.11) and (4.12)

$$\tau^\circ(\alpha_1) = \alpha_1^{-1} I_1(\alpha_1)\left[\frac{\mu}{2\pi} C\left(1 - \frac{\alpha_1^2}{b^2}\right)\right]^{1/2} \tag{4.13}$$

In the general case, performing integration with respect to ϕ in (4.11), we obtain the equation

$$\frac{\pi\tau^{\circ 2}}{\mu[I_1(\alpha)]^2}\left\{\frac{4b^2}{\sqrt{1-\alpha^2 b^{-2}}}\left[\sqrt{1-\alpha^2 a^{-2}}\sqrt{1-\alpha^2 b^{-2}} - \left(1-\frac{\alpha^2}{2b^2}\right)^2\right] + \alpha^2\sqrt{1-\alpha^2 b^{-2}}\right\} = C \tag{4.14}$$

which determines a certain effective value of α. It is apparent that the closer τ° is to the value (4.13), the better is the description provided by this solution for the crack propagation process.

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